

"Science is built up with facts, as a house is with stones. But a collection of facts is no more a science than a heap of stones is a house." Henri Poincare

3 1. (20 pts.) Use the completeness axiom to prove that every monotone decreasing sequence which is bounded from below is convergent.

10 2. (10 pts.) (a) If A, B are two non-empty subsets of \mathbb{R} , prove that $(A \cup B)' = A' \cup B'$. (b) Give the definition of a compact set, and state without proof a theorem that characterizes compact subsets in \mathbb{R} .

15 3. (20 pts.) A sequence a_n is defined by

$$a_1 = 1, a_{n+1} = a_n + \frac{1}{na_n}, n \geq 1.$$

Obtain the asymptotic behavior of a_n as follows: write, for a function f , the continuous analogue of the given recurrence; find f , and use your answer to guess the asymptotic behavior of a_n . Finally, and most importantly, supply a proof of your guess for the asymptotic behavior of the sequence.

4. (15 pts.) The series

$$\sum_{k=1}^{\infty} \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2}\right) \frac{\cos kx}{k},$$

15 where x is a real number, is of the form $\sum_{k=1}^{\infty} a_k \cos kx$. (a) What can you say about a_k ? Justify your answer. (b) Let $C_n(x) = \sum_{k=1}^n \cos kx$. Find and prove a formula for $C_n(x)$, then find all values of x for which the series is convergent and prove your answer. [You may state without proof the theorem needed here].

15 5. (20 pts.) Let k be a positive constant and α the positive root of the quadratic equation $x^2 - x - k = 0$. If β is the second root of the equation what is the sign of β ? Now take a positive number $x_1 < \alpha$, and define a sequence x_n by

$$x_{n+1} = \sqrt{k + x_n}, n \geq 1.$$

(a) What is the sign of $x_1^2 - x_1 - k$? What does it tell about x_1 and x_2 ? (b) Prove that the sequence x_n is convergent and find its limit!

6. (15 pts.) The sequence c_n is defined by

$$c_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln(n+5), n \geq 1.$$

Prove that c_n is a convergent sequence.